ARTICLES

The Earth's radius and the *G* variation

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It has long been assumed that if the gravitational constant G was larger in the past, the Earth's radius had to be smaller. The assertion holds provided the input from microphysics (in particular the equation of state) is independent of G. While this is true for some theories of gravity with variable G it is not so in the scale covariant theory, where the pressure can be affected by a variable G in a way that, for a constant mass of the Earth, a larger G in the past implies a larger Earth's radius. Comparison with recent palaeomagnetic data is presented.

MODERN observations[1] of increasing accuracy seem to confirm that the phenomenon of gravity is correctly understood in terms of Einstein's geometrical interpretation and satisfactorily described by the Einstein equations. There, therefore, seems to be no cogent reason to modify Einstein's theory as a description of gravity, as there is no reason to modify atomic theory. Within their limits, gravitational theory and atomic theory are complete and thus far satisfactory theories.

In cosmology, however, where gravitational phenomena involving large fractions of the age of the Universe are analysed not with gravitational but with atomic clocks, the two dynamics must be coupled. Let Δt_E (E for Einstein) be the time interval marked by a clock governed by gravitational dynamics (two orbiting planets, for example) and Δt the time marked by an atomic clock. Because the gravitational constant G and the total mass M may be thought of as the 'springs' of the gravitational clock, much in the same way as e, $\hbar$ and m may be considered the 'springs' of the atomic clock, then $\Delta t_E = f(G, M)$ and $\Delta t = g(e, \hbar, m)$.

To study the large times involved in cosmology a relationship between Δt_E and Δt as a function of time or equivalently of the age of the Universe is required. The lack of a unified theory of electromagnetic and gravitational forces, which would give us the answer, has been avoided so far by assuming that the ratio $\beta(t) = \Delta t_E/\Delta t$ at any time in the past was the same as today. This is known as the strong equivalence principle, SEP. While the adoption of the SEP has not yet led to flagrant disagreement with observations, it remains an unproved assumption, because its confirmation would require two measurements of $\beta(t)$ at two different times, the age of the Universe serving as a time scale. A more popular way of expressing a possible non-constancy of $\beta(t)$ is by saying that G may vary with respect to atomic time[2]. In fact, a non-constant $\beta(t)$ is equivalent to one of the springs of the gravitational clock, say G, varying with respect to the atomic time.

Given the fundamental role of the SEP for the entire cosmological edifice, it is natural to expect that the nature of the function $\beta(t)$ be determined not by an *a priori* assumption, but by either a unified theory, which we do not have, or by observations of which we now have enough, covering a large fraction of the age of the Universe.

Cosmological data to put limits on $\dot{\beta}/\beta$ were used in ref. 3 while the past luminosity of the Sun was analysed in refs 4 and 5. This article concentrates on the dependence of the radius of the Earth on $G = G(t)$.

The newtonian framework is shown to be unsuitable because it leads to inconsistencies (see refs 3, 4, 6, 7). Then it is shown that the Einstein framework leads, as expected, nowhere. Finally, the scale covariant theory is used through which the effects of a non-constant $\beta(t)$ can be treated and quantified consistently.

The Earth's radius

While the first analysis[8] of the consequences of $G = G(t)$ dealt with the Sun's luminosity (the result $L_\odot \sim G^7$ has been shown[4] to be $L_\odot \sim$ constant; see also ref. 5), more attention seems to have been given to the Earth's radius[9,10]. The general consensus seems to be that if G was larger in the past, the Earth's radius had to be smaller (see, for example, refs 9–12).

To arrive at this qualitative conclusion, one uses (*a*) the hydrostatic equilibrium equation, (*b*) an equation of state and (*c*) the assumption of a constant total rest mass, ($\rho_0 = mn$, $n = N/V$, $M \equiv mN$)

$$\frac{1}{\rho_0}\frac{\mathrm{d}p}{\mathrm{d}r} = -G\frac{m(r)}{r^2} \tag{1a}$$

$$p \propto \rho_0^\gamma \tag{1b}$$

$$M = \text{constant} \tag{1c}$$

thus deriving

$$R \propto M^{(\gamma-2)/(3\gamma-4)} G^{-1/(3\gamma-4)} \sim G^{-1/(3\gamma-4)} \tag{2}$$

which is the basic relation for all the qualitative conclusions that a larger G implies a smaller R, and therefore an expanding Earth.

While equations (1*a*), (1*b*) and (1*c*) form a set of consistent relations when G is constant, is the same true when G becomes a variable? The different exponentials of G and M in equation (2) are in sharp contrast with equation (1*a*) where G enters only in the combination GM. The asymmetry is introduced using an equation of state in the form (1*b*), which contains M but not G, the usual justification being that atomic relations are immune from whatever happens to G, which is valid only if gravity and atomic physics are kept separated. However, a variation of G represents a change of one dynamics with respect to the other, a process that can only be meaningful if we relate, not separate, the two dynamics. While the assumption that the two dynamics are disconnected may be logically consistent with the SEP, the purpose here is to relax the SEP allowing the two dynamics to depend on one another through the function $\beta(t)$ or equivalently $G(t)$. Furthermore, while we may reasonably expect an isolated hydrogen atom to be independent of β, this need no longer be true for a many-body quantity like an equation of state, which involves ingredients like the integration over phase space, the Liouville equation, as well as thermodynamic relations.

To show how equation (1*b*) may lead to incompatibilities with a $G = G(t)$, let us consider the equation of state $pV = NkT$, and use the adiabatic relation $TV^\Gamma = \text{constant}$, where $N = \text{constant}$, that is $M = \text{constant}$. We obtain $p \propto \rho_0^\gamma$ ($\gamma = 1 + \Gamma$), that is equation (1*b*) (equations (1*b*) and (1*c*) are, therefore, related). Furthermore, the basic ingredient in this derivation is the

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adiabatic relation $TV^{\Gamma}=\text{constant}$, which comes from integrating the first law of thermodynamics, which in turn is a statement of conservation of energy.

However, if $G=G(t)$, the potential energy $F=-G(t)m_1m_2/r^2=-\nabla V(r,t)$ becomes time dependent and we do not have energy conservation[13]. An inconsistency has surfaced.

The argument may be formalized by moving from the newtonian framework to the einsteinian one, not because general relativistic effects are relevant to the Earth, but because *GR* is the correct description of gravity. While the previous argument indicates that equations (1*b*) and (1*c*) are related, adopting the einsteinian framework will show that all the relations (1) are connected. In fact, the derivation of equation (1*a*) is based on the momentum conservation law, while equations (1*b*) and (1*c*) are related to the energy conservation law for the total energy density ρ. These laws can be combined into the conservation of the energy momentum tensor $T_{\mu\nu}$ (ref. 14, equation 126.1), which in arbitrary coordinates reads

$$T^{\mu\nu}{}_{;\nu}=0 \tag{3}$$

Let us now consider Einstein's equations,

$$G_{\mu\nu}\equiv R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R; \qquad G_{\mu\nu}=-8\pi GT_{\mu\nu}; \tag{4a}$$

$$G^{\mu\nu}{}_{;\nu}=0 \tag{4b}$$

Equation (4*b*) is the Bianchi identity, a geometrical relation valid irrespectively of the content of the right-hand side of equation (4*a*)[15]. Clearly equations (1), or the global form, equation (3), are compatible with equation (4*b*) only if G is constant.

This argument indicates that taking G in equation (1*a*) variable is inconsistent with the simultaneous use of equations (1*b*) and (1*c*) in which G is absent not because it has no *a priori* right to be there, but because it has already been assumed to be constant.

The Einstein framework

Conventional general relativity is based on the existence of a metric field which obeys the Einstein field equations (4*b*) with or without the right-hand side (ref. 16 and P. Bergmann, personal communication), $GT_{\mu\nu}$, a quantity that Einstein[17] considered "a formal condensation of all those things whose comprehension in the sense of a field theory is still problematic". Equations (4*a*) were constructed from left to right[15]. Equation (4*b*) means that the right-hand side of equations (4*a*) must have zero divergence. While one may choose G to be constant, thus yielding $T^{\mu\nu}{}_{;\nu}=0$ which embodies the 'standard' equations of motion and the 'standard' energy conservation equation, this choice is clearly not dictated by the theory of gravity. For that reason Einstein considered his theory 'incomplete', in the sense that it did not prescribe how to construct the source term from within. Given equation (4*b*), it would have been equally possible to choose

$$(GT^{\mu\nu})_{;\nu}=0 \tag{5}$$

with G as a variable. Equation (5) indicates that $T^{\mu\nu}{}_{;\nu}=0$, equation (3), is no longer true, because there is now a source term proportional to $G_{,\nu}$. Because the energy conservation equation has acquired new terms involving G, so will the resulting equation of state. To summarize, as the energy-momentum tensor $T_{\mu\nu}$ is a function of the pressure p and the total energy density ρ, we have from equation (4) the two possibilities:

G constant plus Bianchi identity $\rightarrow T^{\mu\nu}{}_{;\nu}=0\rightarrow p=p(\rho_0), M=$ constant

G variable plus Bianchi identity $\rightarrow (GT^{\mu\nu})_{;\nu}=0\rightarrow p=p(\rho_0,G)$, $M\neq$ constant

independently of the specific form of $T_{\mu\nu}(p,\rho)$. To illustrate the modification to the p versus ρ relation, consider the following form for $T_{\mu\nu}$

$$T_{\mu\nu}(p,\rho)=(p+\rho)u_\mu u_\nu-pg_{\mu\nu} \tag{6}$$

By introducing the variables $p_*=pG$ and $\rho_*=\rho G$, we formally reduce equation (5) to the standard form $T_*^{\mu\nu}{}_{;\nu}=0$. Because G has 'disappeared', we recover again, as in the standard case $p_*\sim(\rho_0)_*^\gamma$ and $M_*=\text{constant}$. Returning to the variables p and ρ_0, we obtain finally

$$p\propto\rho_0^\gamma G^{\gamma-1} \tag{7a}$$

$$MG=\text{constant} \tag{7b}$$

Alternatively, as $T_*^{\mu\nu}{}_{;\nu}=0$ implies $p_*\mathrm{d}V+\mathrm{d}U_*=0$, on integrating, $(pV=NkT=\Gamma U)$, $NTGV^{\Gamma}=\text{constant}$. However, the zero pressure limit of $T_*^{\mu\nu}{}_{;\nu}=0$ implies $nVG=NG=\text{constant}$, so that substituting T in $pV=NkT$, we recover equation (7*a*) (see also ref. 4).

As for the hydrostatic equilibrium relation, the introduction of the new variables p_* and ρ_* has no effect on its structure, because G depends only on time. Equation (1*a*) is therefore consistent with a varying G.

Inserting equation (7*a*) into equation (1*a*), we derive,

$$R\sim M^{(\gamma-2)/(3\gamma-4)}G^{-1/(3\gamma-4)}G^{(\gamma-1)/(3\gamma-4)}$$
$$\sim(GM)^{(\gamma-2)/(3\gamma-4)}\sim\text{constant} \tag{8}$$

a result radically different from equation (2).

Alternatively, we can say that because only the starred quantities are physically meaningful, equations (1) should read ($\rho_0=\rho$ for ease of notation)

$$\frac{1}{\rho_*}\frac{\mathrm{d}p_*}{\mathrm{d}r}=-\frac{m_*}{r^2}, \qquad p_*\propto\rho_*^\gamma, \qquad M_*=\text{constant} \tag{9}$$

which clearly show that G does not appear.

Scale covariant framework

The previous analysis has indicated that while it is in principle possible formally to make room for a variable G within the Einstein framework, the results turn out to be identical to those with a constant G. The message is that the quantity G has no 'identity' in such framework, thus confirming that there is no useful way of talking about a variable G within the Einstein theory.

To incorporate fully the possibility that atomic and gravitational times may differ over large fractions of the age of the Universe a formalism, the scale covariant theory (SCT) of gravity, was devised. A gauge function $\beta(t)$, epitomizing our ignorance of how gravity and atomic forces couple, was introduced through the equation

$$\Delta t_{\mathrm{E}}=\beta(t)\,\Delta t \tag{10}$$

While the strong equivalence principle assumes $\beta(t)=\text{constant}$, a broad-based analysis of observational data ranging from astrophysics to cosmology (up to $z\approx10^3$), gave no indication that $\beta(t)$ has to be constant[3,4,6,7,18]. While this does not mean that $\beta(t)$ must be variable, it is a significant result once we consider that it was firmly believed that even simple data like the luminosity of the Sun would have clearly opposed the variability of $\beta(t)$. Furthermore, recent measurements of the period of the Moon by both atomic and gravitational clocks do not seem to yield the same result, the residual corresponding to

$$\frac{\dot\beta}{\beta}=(2.75\pm0.64)\,10^{-11}\quad\mathrm{yr}^{-1} \tag{11}$$

Gravitational times are defined as those times with respect to which the Einstein equations retain exactly their form as do all the purely gravitational expressions ensuing therefrom. However, with respect to atomic units, the gravitational equations take the form (ref. 6, equation (2.10))

$$G_{\mu\nu}+f_{\mu\nu}(\beta)=-8\pi GT_{\mu\nu}(\beta) \tag{12}$$

The final aim of the SCT is to construct a viable two-times theory, within which the commonly assumed identity between atomic and gravitational times may be broken, $\Delta t\neq\Delta t_{\mathrm{E}}$. For that to occur, the lagrangian $\mathscr{L}$ describing gravity and matter must not be scale invariant. While it is not *a priori* clear which of the terms in $\mathscr{L}$ should break the symmetry, the choice is not large. Brands and Dicke (BD) chose the gravitational part $\mathscr{L}_g$, leaving $\mathscr{L}_m$, the matter part, unaltered. This implies that their $T_{\mu\nu}$ does not depend on β: the conservation law (3) holds unchanged, as do relations (1). Relation (2) is therefore consistent with the BD framework. However, the change in $\mathscr{L}_g$ means that the Einstein equations are modified by the addition of terms which coincide with our function $f_{\mu\nu}(\beta)$ only if the BD parameter ω is $-3/2$. The potentially most serious source of troubles facing a theory that alters the Einstein equations is that the expressions for the three fundamental tests acquire extra contributions which may spoil the ever increasing agreement with observations. As this seems to be the case in the BD theory, the SCT was constructed with the opposite point of view. The scale breaking terms were relegated entirely to $\mathscr{L}_m$, while the gravitational part of $\mathscr{L}$ was constructed in a scale invariant manner. The result is equation (12), where the function $f_{\mu\nu}(\beta)$ assures that the left-hand side of equation (12) is scale invariant, that is it has power zero under the scale transformation $ds \to ds' = \beta(x)\,ds$, where for any quantity A, we have $A \to A' = \beta^{\pi(A)}A$, $\pi(A)$ being the 'power' of A.

This approach has been questioned[19] on the grounds that the 'presently accepted microscopic lagrangian' can be shown to be scale invariant (a more detailed discussion is given elsewhere). One of the critical quantities entering the analysis of ref. 19 is $\pi(m)$, the power of a microscopic mass m. From the fact that $\hbar/mc$ is a length, it follows that $\pi(l) = \pi(\hbar) - \pi(m) = +1$. The claim[19] that $\pi(m) = -1$ can only be made if $\pi(\hbar) = 0$, which in turn implies $\pi(e) = 0$, because $e^2/\hbar c$ is a pure number. Now because the electric charge enters only in the lagrangian term representing interactions, any assumption about $\pi(e)$ is an assumption about the scaling property of the interaction. Because of the general relation βGM = constant (see equation (14)), $\pi(m) = -1$ implies $g = 2$. Now considering the electromagnetic tensor $F^{\mu\nu}$, if the contribution to $T_{\mu\nu}$ from the free electromagnetic field is taken to be scale invariant then the power of $F^{\mu\nu}$ is $-3-g/2$ which is -4. Because $J^\mu = enu^\mu$ has power -4, the interaction equation $F^{\mu\nu}{}_{;\nu} = J^\mu$ is scale invariant. As such, it may not hold in the SCT because the final aim is that of allowing atomic clocks to scale differently from gravitational clocks. Because the gravitational part $\mathscr{L}_g$ has been constructed in a scale invariant form (hence scale invariant theory of gravitation) with the result that the period of a planet with respect to atomic clocks scales like $P \sim \beta^{-1}$ (ref. 6, equation (4.19)), clearly the dynamic equation governing an atom cannot be $F^{\mu\nu}{}_{;\nu} = J^\mu$ because being scale invariant, it would yield $P(\text{atom}) \sim \beta^{-1}$ too, thus leading to no difference between atomic and planetary dynamics, that is to a constant G. If we accept "the presently accepted form of $\mathscr{L}_m$"[19] (an equation like $F^{\mu\nu}{}_{;\nu} = J^\mu$) then clearly the only possibility for a varying G is to go back to $\mathscr{L}_g$ and to devise a modified BD approach. This line of reasoning, which does not contemplate the possibility advocated by the proponents of the SCT, cannot be taken to imply nor to have proved that either (1) G is constant or (2) that for it to vary, the only avenue open is $\mathscr{L}_g$. Progress in modifying $F^{\mu\nu}{}_{;\nu} = J^\mu$ using gauge fields and neutral current concepts, has recently been made (P.J. Adams and J. Anderson, personal communication). Far from being mutually exclusive, the approach proposed in ref. 19 and the point of view of the SCT are in our opinion but different approaches towards the same goal.

Let us now prove that βGM is constant. It has been shown[6] that the left-hand side of equation (12) has zero co-covariant derivative (indicated by * instead of ;). We therefore have, using equation (A.19) of ref. 6 and the fact that $G_* = 0$,

$$T^{\mu\nu}{}_{;\nu} + (\pi+6)\phi_\nu T^{\mu\nu} - \phi^\mu T^\nu_\nu = 0 \tag{13}$$

with $\phi \equiv \ln\beta$, $\phi_\nu \equiv \phi_{,\nu}$. The quantity π is the power of $T^{\mu\nu}$. Multiplying equation (13) by u_μ and using for $T_{\mu\nu} = \rho_0 u_\mu u_\nu$, we obtain on integrating ($\rho_0 V = M$)

$$\beta GM = \text{constant} \tag{14}$$

where $\dot{V}/V = u^\mu{}_{;\mu}$, and $\pi = \pi_{\text{SI}} = -4-g$, $g = \pi(G)$. We have required that the right-hand side of equation (12) be scale invariant, $\pi(GT_{\mu\nu}) = 0$. Equation (14) makes it clear that the power of m must be chosen consistently with the dynamical constraints of the theory, and not assumed *a priori*. In fact because $\pi(m) = 1-g$, a $G\beta^2 = 1$ gauge yields $\pi(m) = -1$, whereas $G\beta = 1$, the so-called non-matter creation case, yields $\pi(m) = 0$.

The very fact that in equation (12) $T_{\mu\nu}$ is taken to depend on β implies, even without further qualifications, that we cannot use, without checking their validity, any of the standard relations (1*a*), (1*b*) and (1*c*) because, strictly speaking, they are valid only for constant β. The β dependence of microscopic relations, marking a sharp contrast with the BD theory, is the price one must pay for constructing a theory with a built in two-times scheme and which at the same time preserves the agreement between observations and predictions for the fundamental tests, whose expressions in atomic units can be derived by direct scaling of the corresponding gravitational expressions. Because at any given time the scale function $\beta(t)$ can be normalized to unity, a set of measurements at one time cannot reveal the presence of β. In this sense, the SCT theory has preserved agreement with the three tests[7]. The scaling of gravitational expressions to obtain the corresponding atomic ones is all that is required when dealing with purely gravitational phenomena, like periods or distances of planets, which by assumption are given by Einstein equations and are constant with respect to gravitational clocks. On the other hand, non-gravitational quantities must be derived consistently with equation (13), so that the influence of the function β can be fully accounted for. This is a major endeavour because it calls for a reconstruction of microscopic physics, a process which, while accounting for the fact that β can enter at any stage, must satisfy the following two constraints: (1) the atomic clock ensuing from such theory must not depend on β; (2) the many-body behaviour of the theory must yield results consistent with the classical ones derived from equation (13). A typical example of such 'matching' relation is the continuity equation, $\dot{\rho} + \rho \operatorname{div} j = 0$, which can be obtained both from equation (3) and from the Schrödinger equation. These requirements clearly indicate that the attempt at constructing such theory is far more complex and difficult than that faced by Brans and Dicke. Descending from the Einstein equation to microphysics (thus running in the opposite direction to the traditional one, which assumes the validity of the microscopic scheme and makes connection with gravity by going from a flat to curved space) has to my knowledge never been worked out in detail. In spite of this, we have already derived[6] the influence of β on several microscopic relations, and the corresponding observational implications such as the energy versus frequency for a free photon; the adiabatic scaling law for a perfect gas; the equilibrium radiation versus frequency and temperature expression; and the luminosity versus distance relation (see also ref. 5). No contradictions have yet appeared.

Having clarified this point, let us return to the original problem of evaluating R as $R(t)$. The full problem is very complex because we do not yet possess a full description of atomic interactions which determine the equation of state for an object like the Earth. We can therefore present only a hopefully good approximation to the exact solution. Returning to equation (13), and using for $T_{\mu\nu}$ equation (6), we first separate out the rest mass contribution from the total energy density ρ, $\rho = \rho_0 + u$. The rest mass part of equation (13) can be easily integrated. The result is equation (14). The $\rho - \rho_0 = u$ part satisfies the equation ($U = uV$)

$$dU + p\,dV + [(1-g)U + 3pV]\,d\phi = 0 \tag{15}$$

which is the first law of thermodynamics generalized to include the β terms[3]. Let us now write

$$p = \frac{N}{V}kT\left[1 + B(T)\frac{N}{V} + C(T)\left(\frac{N}{V}\right)^2 + \cdots\right] \tag{16}$$

where $B(T)$ and $C(T)$ are the so-called virial coefficients[20]. We first consider the perfect gas case, $B = C = 0$. Using $pV = NkT$

and $U = NkT/\Gamma$ and remembering that $N \sim (\beta G)^{-1}$, we integrate equation (15), with the results ($\gamma = 1+\Gamma$)

$$p \propto \rho_0^{\gamma}\left(\frac{G}{\beta^2}\right)^{\gamma-1} \qquad (17a)$$

$$T\beta^{3\Gamma} \sim \frac{1}{V^{\Gamma}} \qquad (17b)$$

For a more general derivation, valid for non-adiabatic transformations, see ref. 5.

Note that there is no choice of gauge, a G versus β relation, that reduces equations (17a) and (14) to equations (1b) and (1c), at least for $\gamma \neq 1$. In fact, to recover equation (1b), we must choose $G \sim \beta^2$, which implies $\dot\beta < 0$, contrary to equation (11). This choice would also imply $M \sim \beta^{-3}$, not $M \sim$ constant. To see the consequences of equation (17a), we use it in equation (1a), which has already been shown[6] to be valid in the present framework:

$$R \sim M^{(\gamma-2)/(3\gamma-4)} G^{-1/(3\gamma-4)} \left(\frac{G}{\beta^2}\right)^{(\gamma-1)/(3\gamma-4)} \sim \frac{1}{\beta} \qquad (18)$$

on using equation (14). The result is independent of γ. We therefore conclude that

$$\frac{\dot R}{R} = -\frac{\dot\beta}{\beta} = -(2.75 \pm 0.64)10^{-11} \quad \mathrm{yr}^{-1} \qquad (19)$$

where we have used equation (11). The predicted value of R, 400 Myr ago, is therefore ($R_0 = 1$)

$$R = 1.0110 \pm 0.002 \qquad (20)$$

which can be compared with the recent palaeomagnetic result[12]

$$R = 1.020 \pm 0.028 \qquad (21)$$

At first sight, the close agreement can be taken as justification for the approximations made to arrive at equation (18). However, a closer scrutiny revels that the result must be taken with great care. First, as stressed in ref. 12, an equation of state of the form of equation (17a), is not valid at the surface of the planet where $p = 0$ but $\rho_s \neq 0$. The criticism is clearly independent of G and β. More important, however, is that the result $R \sim \beta^{-1}$ implies that the radius of the Earth in einstein units $R_E = \beta R$ is constant. While possible, this result is not obvious because atomic quantities like e and m, which enter the relation p_E as $p_E(\rho_E)$ and therefore R_E, are not constant in einstein units.

Having discussed the limitations of equation (17a), we go back to equation (16) and include the interaction term $B(T)$ in the form $B(T) = -BT^{b-1}$. We simulate in this phenomenological way an attractive potential. Although equation (17b) is valid when $B = 0$, it is used because we are performing an approximation to the first term. Equation (16) then becomes ($\rho_0 = \rho$ for ease of notation)

$$p = A\rho^{\gamma}\left(\frac{G}{\beta^2}\right)^{\gamma-1}\left[1 - \frac{B}{A}\rho^{\tilde\gamma}\left(\frac{G}{\beta^2}\right)^{\tilde\gamma-1}\right] \qquad (22)$$

where $\gamma = 1+\Gamma$, $\tilde\gamma = \gamma_* - \gamma$, $\gamma_* = 2 + (\gamma-1)b$. An equation of state for the Earth of the type (22) (for $G = 1$ and $\beta = 1$) has been proposed by Birch[21] with $\gamma = 7/3$ and $\gamma_* = 5/3$. While equation (17a) allows an exact solution, equation (18), equation (22) does not. For our purposes

$$R \sim \frac{1}{\beta}\left[1 - \frac{B}{A}\bar\rho^{\tilde\gamma}\left(\frac{G}{\beta^2}\right)^{\tilde\gamma-1}\right]^{1/(3\gamma-4)} \qquad (23)$$

where $\bar\rho$ is an average (but time dependent) density. We further derive

$$\frac{\dot R}{R} = -r\frac{\dot\beta}{\beta}; \qquad r - 1 = \frac{(2+g)\alpha}{\mathscr{D} - 3\tilde\gamma\alpha} \qquad (24)$$

where $\mathscr{D} = 3\gamma - 4$, $\alpha = x(1-x)^{-1}$, $x = (B/A)\bar\rho^{\tilde\gamma}\ (G/\beta^2)^{\tilde\gamma-1}$. For the values of γ and γ_* given by Birch, we have $r - 1 = (2+g)\alpha/(3+2\alpha)$, which for $g = 1$ ranges from zero ($\alpha = 0$) to 3/2 ($\alpha = \infty$). The first value corresponds to equation (18); $\alpha = \infty$ implies $x = 1$, that is $\bar\rho = \rho_s$, not acceptable. If p must vanish at $\rho = \rho_s$ then $x = (\bar\rho/\rho_s)^{\tilde\gamma}$. In spite of the large excursion in α, the corresponding variation in $r - 1$ is only from 1 to 1.5. Taking $\alpha = 1$ ($x = 1/2$, corresponding to $\bar\rho = 8\ \mathrm{g\ cm^{-3}}$, $\rho_s = 2.84\ \mathrm{g\ cm^{-3}}$) and $g = 1$, we have $r = 1.6$, so that 400 Myr ago

$$R = 1.018 \pm 0.004 \qquad (25)$$

in better agreement with equation (21). Finally equation (24) may alternatively be written, using equation (14),

$$R \sim (GM)^r, \qquad \frac{\dot R}{R} = r\frac{(\dot{GM})}{GM} \qquad (26)$$

to be compared with equation (2). As r is positive (at least for $g + 2 > 0$) the SCT predicts a result qualitatively different from equation (2): for a constant mass M, a larger G implies a larger radius, not a smaller one.

The method used to arrive at equation (22) is not meant to be a microscopic derivation of the p versus (ρ, β) relation for the Earth, but only an heuristic approach to show how β may enter the problem. Even without β, it is difficult to derive[22] an equation of state for the Earth. Because all the $p = p(\rho)$ relations discussed in ref. 22 contain no β and because $\beta = 1$ corresponds to einstein units, one might be tempted to think that the relations in ref. 22 are actually valid in gravitational units and that a simple scaling could then yield the corresponding atomic relations. This is, however, not the case. Such a procedure would yield a relation where the difference of the power of ρ and that of (G/β^2) is unity, like the first term of equation (22). In turn this yields $r = 1$: it can be seen that $r - 1$ is directly proportional to the difference of these two exponents. Furthermore, $r = 1$ implies $R_E =$ constant producing difficulties already discussed. With the present procedure, this does not happen. The powers of ρ and that of (G/β^2) of the second term of equation (22) do not differ by one but by two. An extra term $(G/\beta^2)^{-1}$ appears which makes $r \neq 1$. This extra factor, which breaks the symmetry that one would invariably get out of a simple scaling process, cannot be arrived at from gravitational units where $\beta =$ constant.

Lunar data

Having derived equation (24) but having no estimates for $\dot\beta/\beta$ from within the theory, we can either use palaeomagnetic data for $\dot R/R$ and thus determine $\dot\beta/\beta$ (and possibly $\dot G/G$), or we evaluate $\dot\beta/\beta$ from other data and then compare the resulting $\dot R/R$ with palaeomagnetic data. The only other data available are the rate of change of the Moon's period measured in both atomic and gravitational units: these are ($P = 2\pi/n$ is the period of revolution of the Moon)

$\dot n_E$ (gravitational time) (arc s cyr^{-2})		$\dot n$ (atomic time) (arc s cyr^{-2})		
-26.0 ± 2.0	ref. 23	-21.4 ± 2.6	ref. 26	
-28.5 ± 3.1	ref. 23	-23.6 ± 1.5	refs 26, 27	
-30.0 ± 3.0	ref. 7	-24.6 ± 3.9	refs 26, 28	(27)
-27.4 ± 3.0	ref. 24			
-30.6 ± 3.1	ref. 25			

where, because of equation (10), $n = \beta n_E$. A least-square fit analysis yields ($\dot\beta/\beta$ in $10^{-11}\ \mathrm{yr}^{-1}$); note that 1 cyr = 10^2 yr

All equations	$\dot n_E = -27.97 \pm 0.78$;	$\dot\beta/\beta = 2.75 \pm 0.64$	(28a)
First $\dot n_E$ deleted	$\dot n_E = -29.11 \pm 0.73$;	$\dot\beta/\beta = 3.41 \pm 0.54$	(28b)
Second $\dot n_E$ deleted	$\dot n_E = -27.87 \pm 0.91$;	$\dot\beta/\beta = 2.69 \pm 0.72$	(28c)

Because the value of $\dot\beta/\beta$ retaining all the eight equations turns out to fall between the other two, we have adopted this value, equation (11). Finally, independently of the way we determine $\dot\beta/\beta$ (palaeomagnetic or lunar data), the value of $\dot G/G$ is gauge dependent and is to be determined using the relations $\dot G/G = -g\dot\beta/\beta$, $\pi(m) = 1 - g$.

Moment of inertia

Using equation (24), and M = constant we obtain $\dot{I}/I = -2r\dot{\beta}/\beta$, or equivalently $\dot{I}/I = 2r\dot{G}/G$, with $r>0$. This result agrees with equation (A.16) of ref. 29, because the G in equation (A.1) must be written as $G_* = G(\beta^2/G)^{\gamma-1} = G^{4-3\gamma}$, due to equation (17$a$). If so, equation (A.16) becomes $\dot{I}/I = a\dot{G}/G$, with $a = (3\gamma-4)/10>0$.

This result is, however, of limited use because it is in atomic units and it represents only the cosmological and not the total contribution to $\dot{I}/I$, whereas the estimates reported in the literature are total variations—due to all possible causes, cosmological and otherwise—and they are all with respect to gravitational units. To prove this last point, we must remember that the basic ingredient (see refs. 30, 31) is the conservation of the total angular momentum for the Earth–Moon system $J_E + L_E$ = constant. If the Earth is considered a point mass, this implies the conservation of the orbital angular momentum L_E, a law valid only with respect to gravitational units[6,7]. In atomic units L scales like $L \sim \beta^2/G$ ($\sim G^{-3}$ for constant mass) (ref. 6, equation 4.7).

We now consider $\dot{I}_E/I_E$ from all sources and expressed in gravitational units. While growth lines in fossils promised to be a valuable source of information[30], recent difficulties in interpretation have indicated that quantitative results are not yet possible[32]. We shall therefore rely on ancient astronomical data (see Chapter 10, ref. 31), and take equation (10.2.2b) from ref. 31 and generalize it to include a non-constant I_E. We obtain

$$\frac{\dot{I}_E}{I_E} = -\frac{\dot{\Omega}_E}{\Omega_E} + \frac{\gamma}{3}\frac{\dot{n}_E}{n_E} + H \tag{29}$$

where $\gamma = (1+b)L_0/J_0$, b accounting for the solar contribution. H represents the contribution of the time variation of dynamical elements describing the size, shape and orientation of the Earth–Moon system. If $J = I\Omega\cos\phi$, and $L = M_e M_m/(M_e+M_m) \times R^2 n(1-e^2)^{1/2}\cos i$, then $H = H(\dot{e}, \dot{\phi}, \mathrm{d}i/\mathrm{d}t)$ (see ref. 33).

Ancient astronomical observations can be used[31] to construct a relation of the form $X_i\dot{n}_E + Y_i\dot{\Omega}_E = f$, where X_i and Y_i are numerical coefficients and f is a function of the observed less the computed position of the recorded event. As stated by Lambeck[31] "the computed positions and locations" are arrived at "assuming purely gravitational motion", further substantiating our claim that the ingredients are the standard newtonian equations which hold only with respect to dynamical units with G_E and M_E constant. Lambeck summarizes the results as follows

$$\dot{n}_E = -28 \pm 2; \qquad \dot{\Omega}_E = -1{,}100 \pm 100 \tag{30}$$

This value for $\dot{n}_E$ is not substantially different from the one determined in equation (28). Using $L_0/J_0 = 4.89$ and $b = 0.21$, we obtain from equation (29)

$$\frac{\dot{I}_E}{I_E} = -(0.086 \pm 0.031)\ 10^{-9}\ \mathrm{yr}^{-1} + H \tag{31}$$

To complete the computation we must now evaluate and then subtract from equation (31) the purely cosmological contribution. This is done by remembering that $I_E \sim M_E R_E^2 \sim \beta^2 R^2$, and using equation (24)

$$\left(\frac{\dot{I}_E}{I_E}\right)_{\mathrm{cosm}} = 2\frac{(\dot{\beta R})}{\beta R} = -2(r-1)\frac{\dot{\beta}}{\beta} = -(0.033 \pm 0.008)\ 10^{-9}\ \mathrm{yr}^{-1} \tag{32}$$

The difference

$$\frac{\dot{I}_E}{I_E} - \left(\frac{\dot{I}_E}{I_E}\right)_{\mathrm{cosm}} = -(0.053 \pm 0.032)\ 10^{-9}\ \mathrm{yr}^{-1} + H \tag{33}$$

may now be attributed to geophysical phenomena such as the formation of the Earth's core[30]. For the evaluation of H see ref. 33.

Conclusions

The question whether a viable theory of gravitation exists that permits equations (1a), (1b) and (1c) as they are, while G is taken to vary, has been investigated with the following results.

Standard Einstein theory can be formally generalized to include a non-constant G. While equation (1a) remains valid, both equations (1b) and (1c) change in such a way that the final result is R = constant, demonstrating that G is actually a non-entity within the strict framework of Einstein.

In the BD theory, as by construction the matter lagrangian is not changed, equations (1) and (2) both hold. However, it is believed that the BD theory faces difficulties, at least in its present form.

Finally, while equation (1a) holds unchanged in the SCT equations (1b) and (1c) cannot be simultaneously satisfied. The most interesting result, however, which transcends the particular example equation (22), is that microphysics may be affected by a variable G. This may seem unusual at first as microphysics contains no G thus creating the impression that it ought to be immune from whatever happens to G. However, relations like the Boltzman equation, the radiative transfer equation, and the Navier–Stokes equations are derivable from and are consistent with the conservation law $T^{\mu\nu}{}_{;\nu} = 0$ for the energy momentum tensor, a law which holds only if G is constant, as equation (13) shows. The absence of G is actually a consequence of having chosen it to be constant.

More general derivations of classical non-gravitational relations (such as, Liouville equation[5], Boltzmann distribution[5], radiative transfer equation[5] and classical thermodynamics[3,6]) in which β does explicitly appear, are possible, if constructed consistently with the general conservation laws that follow from the Einstein equation. For example, an equation of state of the form (17a) with $\gamma = 4/3$ holds[3,6] for radiation (ρ_0 is replaced by the photon number density n_γ). Because $3p_\gamma = \rho_\gamma$ and $n_\gamma T \sim \rho_\gamma$, $\rho_\gamma \sim (\beta^2 G^{-1})T^4$. It then follows that the Sun's luminosity $L_\odot$ scales like $L_\odot \sim \beta^{-1}/k$, where k is the opacity. For constant k and constant rest mass, $L_\odot \sim G$, contrary to the early statements that $L_\odot \sim G^7$ (for details, see refs 4, 5).

Our comparisons of the theoretical predictions with observations must be treated cautiously. The value of $\dot{\beta}/\beta$ derived from lunar data and used to estimate equations (20) and (25) is still subject to uncertainties due to the difficulties in estimating the errors; there are also doubts concerning palaeomagnetic data, or more precisely the assumptions made to arrive at equation (21) (ref. 10; see, however, ref. 34). Finally the evaluation of the function H, equation (33), is still fraught with large errors (see ref. 33).

We prefer, therefore, to focus the dependence of R on G. If we write, in atomic units,

$$\frac{\dot{R}}{R} = \left(\frac{\dot{R}}{R}\right)_{\mathrm{cosm}} + \left(\frac{\dot{R}}{R}\right)_{\mathrm{other\,sources}} \tag{34}$$

the present work predicts a reversal (compared with the standard treatment, equation (2)), of the sign of the first term in equation (34) when written in terms of $\dot{G}/G$, equation (26).

This is in turn due to the new effect represented by the presence of G in the $p = p(\rho)$ relation: the G in the right-hand side of equation (1a) is overcompensated by the $G^{\gamma-1}$ factor in equation (17a), thus causing a larger G to correspond to a larger radius. However, the evaluation of the second term in equation (34) is outside the scope of this article, and we cannot conclude from our work alone that R is actually decreasing.

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1. Will, C. M. in *General Relativity* (eds Hawking, S. W. & Israel, W.) (Cambridge University Press, New York, 1979).
2. Dirac, P. A. M. *Nature* **139**, 323 (1937)
3. Canuto, V. M., Adams, P. J., Hsieh, S.-H. & Tsiang, E. *Astrophys. J. Suppl.* **41**, 243 (1979).
4. Canuto, V. M. & Hsieh, S.-H. *Astrophys. J.* **237**, 613 (1980).
5. Canuto, M. & Hsieh, S.-H. *Astrophys.* (in the press).
6. Canuto, V. M., Adams, P. J., Hsieh, S.-H. & Tsiang, E. *Phys. Rev. D* **16**, 1643 (1977) (Paper I); **41**, 263 (1979) (Paper III); **41**, 301 (1979).
7. Canuto, V. M., Hsieh, S.-H. & Owen, R. J. *Mon. Not. R. astr. Soc.* **188**, 829 (1979).
8. Teller, E. *Phys. Rev.* **73**, 801 (1948).
9. Jordan, P. *The Expanding Earth* (Pergamon, New York, 1971).
10. Carey, S. W. *The Expanding Earth* (Elsevier, Amsterdam, 1976).
11. Wesson, P. S. *Cosmology and Geophysics* (Adam Hilger, Bristol, 1978).
12. McElhinny, M. W., Taylor, S. R. & Stevenson, D. J. *Nature* **271**, 316 (1978).
13. Bishop, W. T. & Landsberg, P. T. *Nature* **264**, 346 (1976).
14. Landau, L. D. & Lifshitz, E. M. *Fluid Mechanics* (Addison-Wesley, Reading, 1959).
15. Dirac, P. A. M. *General Theory of Relativity*, 25–26; 43–45 (Wiley, New York, 1975).
16. Bergmann, P. *Introduction to the Theory of Relativity* (Prentice Hall, Englewood Cliffs, 1960).
17. Einstein, A. in *Autobiographical Notes* (ed. Schilpp, P. A.) 71, (Open Court, Chicago, 1979).
18. Canuto, V. M. & Hsieh, S.-H. *Phys. Rev. Lett.* **44**, 695 (1980).
19. Bekenstein, J. D. & Meisels, A. *Phys. Rev. D* **22**, 1313 (1980).
20. Landau, L. D. & Lifshitz, E. M. *Statistical Physics*, 224 (Pergamon, New York, 1958).
21. Birch, F. *J. geophys. Res.* **57**, 227 (1952).
22. Stacey, F. D., Brennan, B. J. & Irvine, R. D. *Geophys. Surv.* (in the press).
23. Morrison, L. V. & Ward, C. G. *Mon. Not. R. astr. Soc.* **173**, 183 (1975).
24. Goad, C. C. & Douglas, B. C. *J. geophys. Res.* **83**, 2306 (1978).
25. Lambeck, K. *Phil. Trans R. Soc.* **287**, 545 (1977).
26. Van Flandern, T. C. *Astrophys. J.* (in the press).
27. Dickey, J. O., Williams, J. G. & Yoder, C. F. *EOS* **61**, 46, 939 (1980).
28. Calame, D. & Mulholland, J. D. *Science* **199**, 977 (1978).
29. Nordtvedt, K. & Will, C. M. *Astrophys. J.* **177**, 755 (1972).
30. Runcorn, S. K. *Nature* **193**, 311 (1962); **195**, 1248 (1962); **204**, 823 (1964).
31. Lambeck, K. *The Earth's Variable Rotation: Geophysical Causes and Consequences* (Cambridge University Press, 1980).
32. Scrutton, C. T. in *Tidal Friction and the Earth's Rotation* (ed. Broshe, P.) (Springer, New York, 1978).
33. Hipkin, R. G. in *Growth Rhythms and the History of the Earth's Rotation* (eds. Rosenberg, G. D. & Runcorn, S. K.) (Wiley, New York, 1975).
34. Schmidt, P. W. & Clark, D. A. *Geophys. J. R. astr. Soc.* **61**, 95 (1980).

Structure of catabolite gene activator protein at 2.9 Å resolution suggests binding to left-handed B-DNA

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The 2.9 Å resolution crystal structure of Escherichia coli *catabolite gene activator protein (CAP) complexed with cyclic AMP reveals two distinct structural domains separated by a cleft. The smaller carboxy-terminal domain is presumed to bind DNA while the amino-terminal domain is seen to bind cyclic AMP. Model building studies suggest that CAP binds to left-handed B-type DNA, contacting its major groove via two α-helices. It is possible that the CAP conversion of right- to left-handed DNA in a closed supercoil, is what activates transcription by RNA polymerase.*

REPRESSOR and activator proteins that regulate gene expression at the level of transcription recognize specific nucleotide sequences in double-stranded DNA. Although it has been suggested that specific recognition involves α-helices fitting into the major groove of B-DNA[1–3] or anti-parallel β-strands in the minor groove[4], no experimental evidence exists concerning the actual mode of sequence-specific interaction between protein and double-stranded DNA. Also, the molecular mechanism by which gene activators promote the activity of RNA polymerase, thereby switching on genes, remains unknown. We present here the structure determination of a protein that binds in a sequence-specific manner to double-stranded DNA and propose a mechanism by which the catabolite gene activator protein is able to switch on the catabolite-sensitive genes.

The catabolite gene activator protein (CAP), also called the cyclic AMP receptor protein, functions in *Escherichia coli* in the regulation of several catabolite-sensitive gene operons[5,6]. Regulation by CAP is exerted at the transcriptional level with cyclic AMP acting as an allosteric effector. In the presence of a sufficient concentration of intracellular cyclic AMP, cyclic AMP forms a complex with CAP which binds to specific DNA sites near the promoters of several operons and alters the rate of their transcription by RNA polymerase. In the lactose (*lac*)[7] and arabinose[8] operons, the cyclic AMP–CAP complex is a positive regulator—it potentiates transcription. In the galactose operon, the presence of two overlapping promoters for RNA polymerase makes the situation more complex; cyclic AMP–CAP is required for initiation of transcription from one promoter, but inhibits transcription from the other. Thus, in this case it apparently acts as both a positive and negative regulator of operon expression[9].

The active form of CAP is a dimer of identical subunits of molecular weight 22,500 and 201 amino acid residues[10,11]. Proteolytic cleavage studies are consistent with the CAP subunit having two separate structural domains[12]. Results from a variety of techniques[12–17] suggest that CAP undergoes a significant conformational change on binding cyclic AMP.

To develop a structural basis for understanding (1) the cyclic AMP-induced allosteric transition in CAP, (2) the site-specific recognition of DNA by CAP, and (3) the mechanism of transcription activation by CAP, we have initiated crystallographic studies of CAP and its complexes with ligands. In this article we report the structure of the cyclic AMP–CAP complex at 2.9 Å resolution.

Structure determination

CAP protein was purified and crystallized in the presence of 0.5 mM cyclic AMP as previously described[18]. The crystals are orthorhombic, space group $P2_12_12_1$, $a = 46.5$ Å, $b = 97.1$ Å, $c = 105.4$ Å, with one dimeric CAP molecule per asymmetric unit.

Intensities of crystallographic reflections were measured by diffractometer, using the Wyckoff scan algorithm[19]. Friedel pairs of reflections were measured on native crystals and five heavy-atom derivatives; for derivative data sets only ~75% of the total reflections, consisting of those which were most intense in the native data, were measured. Heavy-atom positions were located and refined by conventional methods[20] and the correct enantiomorph of the heavy-atom positions was determined using anomalous difference Fouriers[21]. A summary of heavy-atom derivative preparation and refinement statistics is presented in Table 1. Combined multiple isomorphous replacement and anomalous scattering phases were computed to 2.9 Å resolution and had an average figure of merit of 0.74.

Part of the electron density map is displayed in Fig. 1. Most of the polypeptide backbone is well ordered and can be traced unambiguously in both subunits; however, regions of local disorder are encountered at the amino termini, the carboxy

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